Spectral correspondence theorem for Higgs fields

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Higgs fields

Definition 1

A *Higgs field* is a square matrix that has power series as its elements.

We can define a *Higgs field* as an element of $\phi \in \mathrm{Mat}_{n\times n}([[x]])$.

(Here $\mathbb{C}[[x]] = \left\{ a_0 + a_1 x + a_2 x^2 + \ldots : a_i \in \mathbb{C}, i \geq 0 \right\}$ is a ring, called the ring of the power series in one variable with complex coefficients.)

Ideals

Definition 2

Let R be a ring. An *ideal* of R is a subset $I \subseteq R$ such that

- for all $f, g \in I$, we have $f + g \in I$,
- for all $f \in I$, $g \in R$, we have $fg \in I$.

Note that every ideal of R is a subring of R, but not vice versa.

Factor rings

Definition 3

Let R be a ring, and I be an *ideal* in R. Then we can define the *factor ring* R/I whose elements are equivalence classes for the equivalence relation:

$$x \sim y \iff x - y \in I.$$

For $x \in R$, we denote by $x + I$ the equivalence class of x. The addition and mupltiplication on R/I are defined as follows:

$$(x + I) + (y + I) = (x + y) + I,$$

$$(x + I)(y + I) = (xy) + I.$$

Note that representing an equivalence class as $x + I$ is not unique. It is possible that $x + I = x' + I$, $y + I = y' + I$.

Characteristic polynomial

Definition 4

Let $\phi \in \mathrm{Mat}_{n\times n}(\mathbb{C}[[x]])$ be a *Higgs field*. The *characteristic polynomial* of ϕ is defined as

$$\chi_\phi(x,y) = \det(yI_n - \phi),$$

where I_n is the identity matrix of size n.

E. g. the *characteristic polynomial* of the matrix $\begin{pmatrix} 0 & x^2 & 0 \\ 0 & 0 & x^2 \\ x & 0 & 0 \end{pmatrix}$

is $\chi_\phi(x,y) = y^3 - x^5$.

Spectral ring

Definition 5

The *spectral ring* of ϕ is the *factor ring*

$$R_\phi = \mathbb{C}[[x, y]/(\chi_\phi(x, y)),$$

where $(\chi_\phi(x, y))$ is the *ideal* $\{\chi_\phi(x, y)f, f \in R_\phi\}$, and $\mathbb{C}[[x, y] = \Big\{y^m f_m(x) + y^{m-1} f_{m-1}(x) + ... + y f_1(x) + f_0(x) :$ $f_i \in \mathbb{C}[[x]]\Big\}$.

Spectral modules

Definition 6

Let $\phi \in \mathrm{Mat}_{n\times n}(\mathbb{C}[[x]])$ be a *Higgs field*, and $R_\phi = \mathbb{C}[[x,y]/(\chi_\phi(x,y))$ be its *spectral ring*. The *spectral module* of ϕ is a module M_ϕ over R_ϕ with coordinatewise addition and the action of R_ϕ on M_ϕ defines as follows:

$$x\cdot \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} xf_1 \\ xf_2 \\ \vdots \\ xf_n \end{pmatrix}, \quad y\cdot \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \phi \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} \phi_{11}f_1 + \ldots + \phi_{1n}f_n \\ \phi_{21}f_1 + \ldots + \phi_{2n}f_n \\ \vdots \\ \phi_{n1}f_1 + \ldots + \phi_{nn}f_n \end{pmatrix}.$$

Spectral correspondence: Introduction

Assuming we have a fixed characteristic polynomial χ of ϕ, is there a normal form for ϕ?

E.g. if $\chi(x, y) = y^n - x$, then ϕ is isomorphic to

$$\left(\begin{array}{ccc|c} & & & x \\ \hline 1 & & & \\ & \ddots & & \\ & & 1 & \end{array}\right)$$

How to characterise all non-isomorphic Higgs fields ϕ for a given characteristic polynomial χ?

In order to answer these questions, we will consider the relationship between *Higgs fields* $\phi \in \mathrm{Mat}_{n\times n}(\mathbb{C}[[x]])$ and special modules over $R = \mathbb{C}[[x, y]]/(\chi)$.

Fraction ring

Definition 7

The *fraction ring* $\mathrm{Frac}(R)$ of R is the set of "fractions" $\frac{r}{q}$, $r \in R$, $q \in R \setminus \{0\}$, modulo the equivalence relation

$$\frac{r}{q} \sim \frac{r'}{q'}, \text{ if } rq' = r'q.$$

Every module M over R induces a module $\widetilde{M} = M \otimes_R \mathrm{Frac}(R)$ over $\mathrm{Frac}(R)$. In other words, the elements of $\widetilde{M}$ are linear combinations of pairs $m \otimes \frac{r}{q}$, $m \in M$, $\frac{r}{q} \in \mathrm{Frac}(R)$, modulo the equivalence relations:

$$m_1 \otimes \frac{r}{q} + m_2 \otimes \frac{r}{q} \sim (m_1 + m_2) \otimes \frac{r}{q},$$

$$m \otimes \frac{r}{q} + m \otimes \frac{r'}{q'} \sim m \otimes \left(\frac{r}{q} + \frac{r'}{q'}\right),$$

$$m\, r' \otimes \frac{r}{q} \sim m \otimes r' \frac{r}{q}.$$

Rank one torsion-free

Definition 8

Assume that R has no zero divisors (there are no non-zero elements that divide zero).
A module M over R is called *torsion-free* if $r \cdot m = 0$, for some $r \in R$, $m \in M$, then $r = 0$ or $m = 0$.

Definition 9

A module M over R is said to have *rank one*, if the induced module $\widetilde{M}$ over $\mathrm{Frac}(R)$ is isomorphic to $\mathrm{Frac}(R)$.

Spectral correspondence theorem

Theorem (Beauville-Narasimhan-Ramanan)

Let $\chi \in \mathbb{C}[[x,y]$ be irreducible, $n = \deg_y(\chi)$. Then there is a one-to-one correspondence between the following two sets:

$$\left\{ \begin{array}{c} \textit{torsion-free} \\ \textit{rank one modules} \\ \text{over } R = \mathbb{C}[[x,y]/(\chi) \\ \text{up to isomorphism} \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Higgs fields} \\ \phi \in \mathrm{Mat}_{n\times n}(\mathbb{C}[[x]]) \\ \text{with the characteristic} \\ \text{polynomial } \chi \text{ up to} \\ \text{conjugation via} \\ \text{elements of} \\ \mathrm{GL}_n(\mathbb{C}[[x]]) \end{array} \right\}$$

Example

Consider the characteristic polynomial $\chi = y^2 - x^k$ (k is odd). Our goal is to describe all torsion-free rank 1 modules over $R = \mathbb{C}[[x, y]]/(\chi)$.

We will need to use the following lemma:

Lemma

For every torsion-free rank one module M over R there is a injective module homomorphism of M into $\overline{R}$ (where $\overline{R} = C[[t]]$) so that:

$$R \subseteq M \subseteq \overline{R}$$

In our example, we can use $x \to t^2$ and $y \to t^k$.

Example

1. $M = R = \mathbb{C}[[x, y]]/\chi$.
Let $(1, y)$ be the basis over $\mathbb{C}[[x]]$. Then, all elements of M can be presented as $f \cdot 1 + g \cdot y$, where $f, g \in \mathbb{C}[[x]]$. In other words,

$$M = \begin{matrix} \mathbb{C}[[x]] \\ \oplus \\ \mathbb{C}[[x]] \end{matrix} \qquad \text{as } \mathbb{C}[[x]]\text{-module}$$

Then, as multiplication by ϕ is the same as multiplication by y,

$$\phi \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \phi \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x^k \\ 0 \end{pmatrix}$$

Hence, in this case $\phi = \begin{pmatrix} 0 & x^k \\ 1 & 0 \end{pmatrix}$.

Example

2. $M = \overline{R}$.

Let $x \cdot m = t^2 \cdot m$, and $y \cdot m = t^k \cdot m$ for all $m \in M$. Then, $(1, t)$ is the basis of M over $\mathbb{C}[[t]]$, and

$$y \cdot 1 = t^k = 0 \cdot 1 + t^{k-1} \cdot t \qquad y \cdot t = t^{k+1} = t^{k+1} \cdot 1 + 0 \cdot t$$

Note that χ is irreducible, so k must be odd. Therefore, we get

$$\phi \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ t^{k-1} \end{pmatrix} = \begin{pmatrix} 0 \\ x^{\frac{k-1}{2}} \end{pmatrix} \qquad \phi \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} t^{k+1} \\ 0 \end{pmatrix} = \begin{pmatrix} x^{\frac{k+1}{2}} \\ 0 \end{pmatrix}$$

Hence, here we get $\phi = \begin{pmatrix} 0 & x^{\frac{k+1}{2}} \\ x^{\frac{k-1}{2}} & 0 \end{pmatrix}$.

Example

Now, recall that $R \subseteq M \subseteq \overline{R}$, and think of M/R as a submodule of $\overline{R}/R =< t, t^3, t^5, ..., t^{k-2} >$.

Let i be the smallest integer such that $t^{2i-1} \in M$.
Then, $t^{2j-1} = t^{2i-1} \cdot x^{j-i}$ $(j > i)$, must also be in M.

Thus, we have shown that $M/R =< t^{2i-1}, t^{2i+1}, ..., t^{k-2} >$.
(Notice that $M = \overline{R}$ is a subcase of the ring above.)

Finally, consider the general case.

Example

3. $M/R = < t^{2i-1}, t^{2i+1}, ..., t^{k-2} >$.
In this case, $(1, t^{2i-1})$ is the basis of M over $\mathbb{C}[[x]]$, and we have:

$$y{\cdot}1 = t^k = 0{\cdot}1 + t^{k+1-2i}{\cdot}t^{2i-1}, \quad y{\cdot}t^{2i-1} = t^{k+2i-1} = t^{k+2i-1}{\cdot}1 + 0{\cdot}t^{2i-1}$$

Therefore,

$$\phi\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\t^{k+1-2i}\end{pmatrix} = \begin{pmatrix}0\\x^{\frac{k+1-2i}{2}}\end{pmatrix}, \ \phi\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}t^{k+2i-1}\\0\end{pmatrix} = \begin{pmatrix}x^{\frac{k+2i-1}{2}}\\0\end{pmatrix}$$

So, the Higgs field $\phi = \begin{pmatrix}0 & x^{\frac{k+2i-1}{2}}\\ x^{\frac{k+1-2i}{2}} & 0\end{pmatrix}$ for $i = 1, 2, ..., \frac{k-1}{2}$.

In conclusion, we have proved that $\phi = \begin{pmatrix}0 & x^{\frac{k-1+2i}{2}}\\ x^{\frac{k+1-2i}{2}} & 0\end{pmatrix}$ for $i = 1, 2, ..., \frac{k+1}{2}$.

Further examples

Higgs fields $\phi \in \mathrm{Mat}_{3\times 3}(\mathbb{C}[[x]])$

- $\chi(x,y) = y^3 - x^4$
- $\chi(x,y) = y^3 - x^5$
- $\chi(x,y) = y^3 - x^7$

Thank you

THANK YOU!